General formalism for the spectral decomposition of the Hamiltonian in a quantum fractal network

General formalism for the spectral decomposition of the Hamiltonian in a quantum fractal network

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Abstract. We present a general formulation for the spectral decomposition of the Hamiltonian operator of a quantum fractal network (QFN). The QFN can be constructed by placing artificial neurons on each site of the fractal lattice. An artificial neuron may consist of a cell of a quantum cellular automata or a quantum dot which confines a single electron. The Coulomb interaction or the spin–spin interaction between neurons can be used to transmit signals and perform logic operations. The local external field may be as input signal to influence output of the system. We obtain explicitly the recursive formulae of the eigenvalues and eigenvectors between sublattices, the intertwining relation between the collision operator and the Hamiltonian operator by combining subdynamics and a reduced lattice approach. Furthermore, the perturbation method to obtain the spectral decomposition for the time-dependent Hamiltonian is also discussed. Finally, as an example, we calculate the eigenvalues and eigenvectors of the Hamiltonian operator for a Sierpinski gasket based on our formulation. Analysis of the recursive formula for the spectrum of the Sierpinski gasket, reveals how its spectral structure changes in boundary conditions.

1. Introduction

The self-organized response of neurons to external stimuli can result in fractal-like patterns. For example, the human brain may assume a fractal-like structure in which neural structures at many different spatial scales are recursively embedded by self-similarity [1]. Fractal structures may, in principle, be fabricated on semiconductor surfaces at the nanometre scale. An artificial neuron may be placed on each lattice point of this fractal net, with each neuron having a cellular structure [2] or being a quantum dot confining a single electron with an arbitrary spin state [3]. The Coulomb interaction or the spin–spin interaction can act as the switching mechanism between neurons. This interaction can be used to transmit signals and perform logic operations. Furthermore, a single neuron with a given spin may be used to construct a quantum logic gate which can convert input qubits into output qubits [4]. An interesting feature of this sort of artificial fractal circuit is that it can produce a complicated electronic spectral structure, quite different from that of a regular neural network [5]. Based on this structure, the physical properties of quantum fractal circuits can be explored and their applications can be examined.

In this paper, we present a formulation for dealing with the spectral decomposition of the time-independent or time-dependent Hamiltonian operator of a fractal lattice by combining subdynamics with a reduced sublattice approach. In section 2, we introduce the collision operator by dividing the system into two sublattices based on the self-similarity of the system. In section 3, we deduce the recursive formulae based on the self-similarity

between the two sublattices of the system. In section 4, we investigate the intertwining relationship between the original Hamiltonian operator and the intermediate operator based on the subdynamics approach. In section 5, the perturbation method to solve eigenvalue problems of the Hamiltonian influenced by a time-dependent local external field is also discussed. In section 6, we demonstrate one application of this approach to construct the eigenvalues and the eigenvectors of the Hamiltonian operator of the Sierpinski gasket [6]. Finally, we provide some conclusions of this investigation.

2. Collision operator of the model

Let us consider a fractal lattice in which an artificial neuron hosting electrons with arbitrary polarization or spin states is placed on each site [2, 7–9]. The Hamiltonian operator H of this confinement quantum fractal system can be written as a bulk part H_b plus a perturbation part ΔH introduced by confinement of artificial neurons:

$$H = H_b + \Delta H. \tag{1}$$

Then, we choose the eigenvectors of neurons of the confinement system as the initial basis to expand the Hamiltonian H by

$$H = \sum_j \alpha_{jj} |j\rangle\langle j| - \sum_{j,k} \beta_{jk} |j\rangle\langle k| + \beta_{kj} |k\rangle\langle j| \tag{2a}$$

where $|j\rangle$, $\langle k|$ represent eigenvectors of neighbouring neurons with the set of all these wavefunctions comprising a complete orthonormal basis in a Hilbert space; the coefficient $\alpha_{jj} = \langle j|H|j\rangle$ is the on-site potential at sites j; $\beta_{jk} = \langle j|H|k\rangle (\beta_{kj} = \langle k|H|j\rangle)$ is the jump potential between the jth (kth) neuron and kth (jth) neuron.

For simplicity, we assume that the shape, size and structure, etc of each neuron for this system is the same so that the confinement is the same for each neuron; the inter-neuron spacing is sufficiently close so that only nearest-neighbour electrons have an appreciable overlap between their wavefunctions, which means that the Coulomb interaction or the spin–spin interaction only influences nearest neighbours. Under these assumptions, the Hamiltonian (2a) becomes:

$$H = \alpha \sum_j |j\rangle\langle j| - \beta \sum_{\langle jk \rangle} (|j\rangle\langle k| + |k\rangle\langle j|). \tag{2b}$$

From Schrödinger's equation

$$i\frac{\partial}{\partial t}\varphi(t) = H\varphi(t) \tag{3}$$

the formal solution is given by

$$\varphi(t) = e^{-itH}\varphi(0) = \frac{1}{2\pi i} \oint_\Gamma dz\, e^{-izt} R(z)\varphi(0) \tag{4}$$

where the contour integral Γ runs anticlockwise around a sufficiently large portion of the real axis of the complex plane; $R(z)$ is the resolvent operator defined by

$$R(z) \equiv \frac{1}{z - H}. \tag{5}$$

Let us divide this system into two parts by introducing two kinds of sublattices based on the self-similarity of the system. That is, sublattice 1 may be deduced from sublattice 2 by

a spatial scaling, and both constitute a certain-order lattice. Then, two projection operators corresponding to the two sublattices are defined by

$$P \equiv \sum_{j=1}^{N^{(1)}} |j^{(1)}\rangle\langle j^{(1)}| \qquad Q \equiv \sum_{k=1}^{N^{(2)}} |k^{(2)}\rangle\langle k^{(2)}| \tag{6}$$

for $j^{(1)} \neq k^{(2)}$ and satisfy completeness in Hilbert space:

$$P + Q = I \tag{7}$$

so, the orthonormal property is automatic:

$$PQ = P(I - P) = 0 \tag{8}$$

where $N^{(1)}$ and $N^{(2)}$ denote the numbers of sites of two sublattices, respectively.

The Hamiltonian operator H and resolvent operator $R(z)$ can then be written:

$$H = \begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix} \tag{9}$$

$$R(z) = \begin{pmatrix} PRP & PRQ \\ QRP & QRQ \end{pmatrix}. \tag{10}$$

In terms of equation (5) we have

$$zPRP - PHPRP - PHQRP = P \tag{11a}$$

$$zQRP - QHPRP - QHQRP = 0 \tag{11b}$$

$$zPRQ - PRPHQ - PRQHQ = 0 \tag{11c}$$

$$zQRQ - QRPHQ - QRQHQ = Q. \tag{11d}$$

Using equation (11b), we get

$$QRP = \frac{1}{z - QHQ} QHPRP. \tag{12}$$

Substituting equation (12) into equation (11a),

$$PRP = \frac{1}{z - PHP - \psi(z)} P \tag{13}$$

where

$$\psi(z) \equiv PHQ \frac{1}{z - QHQ} QHP. \tag{14}$$

Therefore equation (12) becomes

$$QRP = C(z) \frac{1}{z - PHP - \psi(z)} P \tag{15}$$

where

$$C(z) \equiv \frac{1}{z - QHQ} QHP. \tag{16}$$

In the same way, from equations (11c) and (11d) we have

$$PRQ = P \frac{1}{z - PHP - \psi(z)} D(z) \tag{17}$$

$$QRQ = C(z) \frac{1}{z - PHP - \psi(z)} D(z) + Q \frac{1}{z - QHQ} \tag{18}$$

where

$$D(z) \equiv PHQ\frac{1}{z - QHQ}.$$ (19)

Thus,

$$PRQ + QRQ = RQ$$
$$= (P + C(z))\frac{1}{z - PHP - \psi(z)}D(z) + Q\frac{1}{z - QHQ}$$ (20)

and

$$PRP + QRP = (P + C(z))\frac{1}{z - PHP - \psi(z)}P = RP.$$ (21)

Finally, we obtain an identity relation [10, 11]

$$R(z) = R(z)Q + R(z)P$$
$$= (P + C(z))\frac{1}{z - \Psi(z)}(P + D(z)) + Q\frac{1}{z - QHQ}$$ (22)

where we define the collision operator $\Psi(z)$ by

$$\Psi(z) \equiv PHP + \psi(z) = PHP + PHC(z)P$$
$$= PHP + PHQ\frac{1}{z - QHQ}QHP.$$ (23)

The matrix representation of the collision operator is given by

$$\Psi_M = H^{(11)} + H^{(12)}\frac{1}{ZI - H^{(22)}}H^{(21)}$$ (24)

where $H^{(11)}$ is a matrix given with respect to the vectors of the sites of sublattice 1, reflecting the interactions between the sites of sublattice 1; $H^{(22)}$ is a matrix with respect to the vectors of the sites of sublattice 2, reflecting the interaction between the sites of sublattice 2; $H^{(12)}$, $H^{(21)}$ are matrices with respect to the vectors of the sites of sublattices 1 and 2, reflecting the interactions between the different sublattices 1 and 2.

3. Recursive relation for sublattices

The important characteristic of the fractal lattices is self-similarity between sublattices. Based on this fact, we can obtain the recursive relations of the energy spectrum and eigenstates between different-order sublattices, which permit finding the spectral decomposition of the original Hamiltonian operator. The recursive relation is induced as follows.

First, by means of the property of self-similarity between sublattices 1 and 2, we obtain the recursive relations of the eigenvalues between the two sublattices. In fact, from definition (23), we have

$$\Psi(z) = \sum_{j^{(1)}=1}^{N^{(1)}} \left(\langle j^{(1)}|H|j^{(1)}\rangle + \langle j^{(1)}|HQ\frac{1}{z - QHQ}QH|j^{(1)}\rangle \right) |j^{(1)}\rangle\langle j^{(1)}|$$
$$+ \sum_{j^{(1)}\neq k^{(1)}}^{N^{(1)}} \left(\langle j^{(1)}|H|k^{(1)}\rangle + \langle j^{(1)}|HQ\frac{1}{z - QHQ}QH|k^{(1)}\rangle \right) |j^{(1)}\rangle\langle k^{(1)}|.$$ (25)

Since

$$
\langle j^{(1)}|HQ\frac{1}{z-QHQ}QH|j^{(1)}\rangle
$$

$$
= \sum_{k^{(2)},\, j^{(2)}\neq j^{(1)}}^{N^{(2)}} \langle j^{(1)}|H|k^{(2)}\rangle\langle k^{(2)}|\frac{1}{z-QHQ}|j^{(2)}\rangle\langle j^{(2)}|H|j^{(1)}\rangle
$$

$$
= (N'^{(2)}-1)N'^{(2)}\langle k^{(2)}|\frac{\beta^2}{z-QHQ}|j^{(2)}\rangle \tag{26}
$$

and

$$
\langle j^{(1)}|HQ\frac{1}{z-QHQ}QH|k^{(1)}\rangle
$$

$$
= \sum_{k^{(2)}\neq j^{(1)}}^{N^{(2)}} \sum_{j^{(2)}\neq k^{(1)}}^{N^{(2)}} \langle j^{(1)}|H|k^{(2)}\rangle\langle k^{(2)}|\frac{1}{z-QHQ}|j^{(2)}\rangle\langle j^{(2)}|H|k^{(1)}\rangle
$$

$$
= (N'^{(2)}-1)^2\langle k^{(2)}|\frac{\beta^2}{z-QHQ}|j^{(2)}\rangle \tag{27}
$$

where $N'^{(2)}$ is the number of sites on sublattice 2 (which are nearest neighbours of the sites of sublattice 1). Thus,

$$
\Psi(z) = \alpha' \sum_{j^{(1)}=1}^{N^{(1)}} |j^{(1)}\rangle\langle j^{(1)}| + \beta' \sum_{j^{(1)}\neq k^{(1)}}^{N^{(1)}} |j^{(1)}\rangle\langle k^{(1)}| \tag{28a}
$$

giving,

$$
\alpha' = \alpha + (N'^{(2)}-1)N'^{(2)}\langle k^{(2)}|\frac{\beta^2}{z-QHQ}|j^{(2)}\rangle \tag{28b}
$$

and

$$
\beta' = \beta + (N'^{(2)}-1)^2\langle k^{(2)}|\frac{\beta^2}{z-QHQ}|j^{(2)}\rangle. \tag{28c}
$$

Therefore, any diagonal element of the matrix Ψ_M represents α', while any off-diagonal element of Ψ_M represents β'.

Supposing that the eigenvector of H is given by

$$
|f\rangle = \sum_{j=1}^{N^{(1)}+N^{(2)}} \langle j|f\rangle|j\rangle = \sum_{j=1}^{N^{(1)}+N^{(2)}} f_j|j\rangle \tag{29}
$$

then we have

$$
\langle j|H|f\rangle = Z_j\langle j|f\rangle = \alpha\langle j|f\rangle + \beta\sum_k \langle k|f\rangle \tag{30}
$$

$$
(Z_j-\alpha)f_j = \beta\sum_k f_k. \tag{31}
$$

If $\Psi(z)$ and H have the same spectral structure, from equation (28a), we also have

$$
\langle j|\Psi(z)|\varphi^{(1)}\rangle = Z_j\langle j|\varphi^{(1)}\rangle = \alpha'\langle j|\varphi^{(1)}\rangle + \beta'\sum_k \langle k|\varphi^{(1)}\rangle \tag{32}
$$

$$
(Z_j-\alpha')\varphi_j^{(1)} = \beta'\sum_k \varphi_k^{(1)}. \tag{33}
$$

In terms of the self-similarity of the sublattice, comparing equations (31) and (33),

$$\frac{Z_j - \alpha' + a}{\beta'} = \frac{Z'_j - \alpha + a}{\beta'} \tag{34}$$

hence, the recursive formula between the eigenvalues of sublattice 1 and sublattice 2 is given by

$$\varepsilon' = \left(\frac{\beta}{\beta'} \varepsilon - \frac{\alpha' + \alpha}{\beta'} \right) \tag{35}$$

where we have defined

$$\varepsilon' \equiv \frac{Z'_j}{\beta'} \qquad \varepsilon \equiv \frac{Z_j}{\beta}. \tag{36}$$

Secondly, from equation (9), we have

$$
\begin{aligned}
HF &= ZF \\
&= \begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix} \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} \\
&= ZI \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}.
\end{aligned}
\tag{37}
$$

Therefore,

$$PHPu^{(1)} + PHQu^{(2)} = Zu^{(1)} \tag{38a}$$
$$QHPu^{(1)} + QHQu^{(2)} = Zu^{(2)}. \tag{38b}$$

Using equation (38b), we obtain the recursive relation between the eigenvectors of sublattice 1 and sublattice 2:

$$u^{(2)} = \frac{1}{Z - QHQ} QHPu^{(1)} \tag{39}$$

with the matrix representation given by

$$u^{(2)} = \frac{1}{ZI - H_{22}} H_{21} u^{(1)}. \tag{40}$$

4. Intertwining relation

The eigenvalue problem of the collision operator $\Psi(z)$ is closely related to the eigenvalue problem of the original Hamiltonian operator H. To prove this, suppose that

$$\Psi(Z_n^{(v)})|u_n^{(v)}\rangle = Z_n^{(v)}|u_n^{(v)}\rangle \tag{41}$$
$$\langle \tilde{v}_n^{(v)}|\Psi(Z_n^{(v)}) = \langle \tilde{v}_n^{(v)}|Z_n^{(v)} \tag{42}$$
$$P^{(v)} = \sum_{n=1}^{N^{(v)}} |u_n^{(v)}\rangle\langle \tilde{v}_n^{(v)}| \qquad Q^{(v)} = I - P^{(v)} \tag{43}$$
$$\langle u_n^{(v)}|\tilde{v}_m^{(v)}\rangle = \delta_{n,m} \tag{44}$$

where $v = 1, 2$ corresponds to sublattice 1 and sublattice 2, respectively. $\tilde{v}_n^{(v)}$ represents the left eigenvector of $\Psi^{(v)}$ which may be in the functional space beyond Hilbert space [11, 12]. Then, from equation (22), we have

$$
\begin{aligned}
R(z) = \sum_{v=1}^{2} \sum_{n=1}^{N^{(v)}} \Bigg[& (P^{(v)} + C^{(v)}(z)) \frac{1}{z - Z_n^{(v)}} |u_n^{(v)}\rangle\langle \tilde{v}_n^{(v)}| (P^{(v)} + D^{(v)}(z)) \\
& + Q^{(v)} \frac{1}{z - Q^{(v)}HQ^{(v)}} \Bigg].
\end{aligned}
\tag{45}
$$

From equation (3), the spectral decomposition of the evolution operator is calculated by

$$
e^{-itH} = \frac{1}{2\pi i} \oint_\Gamma dz\, e^{-izt} R(z)
$$

$$
= \sum_{v=1}^{2} \sum_{n=1}^{N^{(v)}} \left[\frac{1}{2\pi i} \oint_\Gamma dz\, e^{-izt} (P^{(v)} + C^{(v)}(z)) \frac{1}{z - Z_n^{(v)}} |u_n^{(v)}\rangle\langle \tilde{v}_n^{(v)}| (P^{(v)} + D^{(v)}(z)) \right.
$$

$$
\left. + \frac{1}{2\pi i} \oint_\Gamma dz\, e^{-izt} Q^{(v)} \frac{1}{z - Q^{(v)} H Q^{(v)}} \right] \tag{46}
$$

where Γ is a contour of Cauchy integral including the eigenvalues of H as poles. Suppose that the spectral decomposition of the Hamiltonian operator H is given by

$$
H = \sum_{v=1}^{2} \sum_{n=1}^{N^{(v)}} Z_n^{(v)} |F_n^{(v)}\rangle\langle \tilde{F}_n^{(v)}| \tag{47}
$$

then, since

$$
z - Q^{(v)} H Q^{(v)} = z - \sum_{v \neq 1 n}^{2} \sum_{j,k=1}^{N(v)} Z_n^{(v)} \langle \tilde{v}_j^{(v)} | F_n^{(v)}\rangle\langle \tilde{F}_n^{(v)} | u_k^{(v)}\rangle | u_j^{(v)}\rangle\langle \tilde{v}_k^{(v)}| \tag{48}
$$

therefore the contour integral Γ excludes the eigenvalues of $Q^{(v)} H Q^{(v)}$ as poles, so that

$$
\frac{1}{2\pi i} \oint_\Gamma dz\, e^{-izt} Q^{(v)} \frac{1}{z - Q^{(v)} H Q^{(v)}} = 0. \tag{49}
$$

Then, we obtain

$$
e^{-itH} = \sum_{v=1}^{2} \sum_{n=1}^{N^{(v)}} \frac{1}{2\pi i} \oint_\Gamma d z\, e^{-izt} (P^{(v)} + C^{(v)}(z)) \frac{1}{z - Z_n^{(v)}} |u_n^{(v)}\rangle\langle \tilde{v}_n^{(v)}| (P^{(v)} + D^{(v)}(z))
$$

$$
= \sum_{v=1}^{2} \sum_{n=1}^{N^{(v)}} e^{-iZ_n^{(v)} t} \eta (P^{(v)} + C^{(v)}(Z_n^{(v)})) |u_n^{(v)}\rangle\langle \tilde{v}_n^{(v)}| (P^{(v)} + D^{(v)}(Z_n^{(v)})) \tag{50}
$$

where the normalization factor is given by

$$
\eta = (\langle \tilde{v}_n^{(v)} | (P^{(v)} + D^{(v)}(Z_n^{(v)}) C^{(v)}(Z_n^{(v)})) | u_n^{(v)}\rangle)^{-1}. \tag{51}
$$

By taking the derivative of equation (50), the spectral decomposition of H is:

$$
H = \sum_{v=1}^{2} \sum_{n=1}^{N^{(v)}} Z_n^{(v)} \eta (P^{(v)} + C^{(v)}(Z_n^{(v)})) |u_n^{(v)}\rangle\langle \tilde{v}_n^{(v)}| (P^{(v)} + D^{(v)}(Z_n^{(v)}))
$$

$$
= \sum_{v=1}^{2} \sum_{n=1}^{N^{(v)}} Z_n^{(v)} |F_n^{(v)}\rangle\langle \tilde{F}_n^{(v)}| \tag{52}
$$

where the eigenvalues of the Hamiltonian operator H are the same as the eigenvalues of the collision operator $\Psi^{(v)}(z)$, while the eigenvectors of the Hamiltonian operators are given by

$$
|F_n^{(v)}\rangle = \sqrt{\eta} (P^{(v)} + C^{(v)}(Z_n^{(v)})) |u_n^{(v)}\rangle \tag{53a}
$$

$$
\langle \tilde{F}_n^{(v)}| = \langle \tilde{v}_n^{(v)} | (P^{(v)} + D^{(v)}(Z_n^{(v)})) \sqrt{\eta}. \tag{53b}
$$

Furthermore, by defining the global collision operator Θ [13] as

$$
\Theta^{(v)} = \sum_{n=1}^{N^{(v)}} \Psi(Z_n^{(v)}) |u_n^{(v)}\rangle\langle \tilde{v}_n^{(v)}| = \sum_{n=1}^{N^{(v)}} Z_n^{(v)} |u_n^{(v)}\rangle\langle \tilde{v}_n^{(v)}| \tag{54}
$$

and the global creation operator $C^{(\nu)}$ [13] as

$$C^{(\nu)} = \sum_{n=1}^{N^{(\nu)}} C^{(\nu)}(Z_n^{(\nu)}) |u_n^{(\nu)}\rangle \langle \tilde{v}_n^{(\nu)}| \tag{55}$$

and considering the definition of the collision operator given in equation (23), we have

$$\Theta^{(\nu)} = \sum_{n=1}^{N^{(\nu)}} (P^{(\nu)} H P^{(\nu)} + P^{(\nu)} H C^{(\nu)}(Z_n^{(\nu)}) P^{(\nu)}) |u_n^{(\nu)}\rangle \langle \tilde{v}_n^{(\nu)}|$$

$$= P^{(\nu)} H P^{(\nu)} + P^{(\nu)} H C^{(\nu)} P^{(\nu)} \tag{56}$$

and

$$\Theta^{(\nu)} |u_n^{(\nu)}\rangle = Z_n^{(\nu)} |u_n^{(\nu)}\rangle \tag{57a}$$

$$\langle \tilde{v}_n^{(\nu)}| \Theta^{(\nu)} = \langle \tilde{v}_n^{(\nu)}| Z_n^{(\nu)}. \tag{57b}$$

This means that $|u_n^{(\nu)}\rangle$ and $\langle \tilde{v}_n^{(\nu)}|$ are right and left eigenvectors of the global collision operator $\Theta^{(\nu)}$, respectively. From the definition of the global collision operator (i.e. equation (54)) and the spectral decomposition (i.e. equation (52)), we can obtain an intertwining relation between the original Hamiltonian operator and the intermediate operator by

$$H = \Omega \Theta \Omega^{-1} \tag{58}$$

where the intermediate operator [10–12] is defined by

$$\Theta \equiv \sum_{\nu=1}^{2} \Theta^{(\nu)} \tag{59}$$

the similarity operator is defined by

$$\Omega \equiv \sum_{\nu=1}^{2} (P^{(\nu)} + C^{(\nu)}) \tag{60a}$$

and

$$\Omega^{-1} \equiv \sum_{\nu=1}^{2} (P^{(\nu)} + D^{(\nu)} C^{(\nu)})^{-1} (P^{(\nu)} + D^{(\nu)}). \tag{60b}$$

5. The fractal neural network driven by an external field

When the fractal neural network is subject to a local time-dependent external field $V(t)$ as an input signal, the Hamiltonian (1) becomes:

$$H(t) = H + V(t). \tag{61}$$

In this case, we can choose the eigenvectors ($|\varphi_\kappa\rangle$, $\langle \varphi_\kappa|$) of H as the initial basis, and define the one-dimensional projectors by $P_\kappa = |\varphi_\kappa\rangle \langle \varphi_\kappa|$, and its orthogonal projectors by $Q_\kappa = I - P_\kappa$, where κ may imply the degeneracy parameter. Then the collision operator can be written as:

$$\Psi_\kappa(z) = P_\kappa H(t) P_\kappa + P_\kappa H(t) Q_\kappa \frac{1}{z - Q_\kappa H(t) Q_\kappa} Q_\kappa H(t) P_\kappa$$

$$= P_\kappa H(t) P_\kappa + P_\kappa H(t) R_Q(z) H(t) P_\kappa \tag{62}$$

where $R_{Q_\kappa}(z)$ are denoted as the resolvents in the subspace $\mathcal{H}_{Q_\kappa} = \{Q_\kappa f, \forall f \in \mathcal{H}\}$ of the Hilbert space $\mathcal{H}$ by

$$R_{Q_\kappa}(z) \equiv Q_\kappa \frac{1}{z - Q_\kappa H(t) Q_\kappa} Q_\kappa. \tag{63}$$

From the Lippmann–Schwinger equation and its iterations we have the Borns series

$$R(z) = R_0(z) + R_0(z)V(t)R_0(z) + R_0(z)V(t)R_0(z)V(t)R_0(z) + \cdots \qquad (64)$$

where the resolvents are defined by

$$R(z) \equiv \frac{1}{z - H(t)} \qquad R_0(z) \equiv \frac{1}{z - H}. \qquad (65)$$

By adding the term λW to the $H(t)$ and taking the limit $\lambda \to \infty$, we have

$$\tilde{R}(z) = \tilde{R}_0(z) + \tilde{R}_0(z)V(t)\tilde{R}_0(z) \qquad (66)$$

and can prove that [17]

$$\begin{aligned}
R_{Q_\kappa}(z) = \tilde{R}(z) &= \lim_{\lambda \to \infty}(z - H(t) - \lambda W)^{-1} \\
&= Q_\kappa \frac{1}{z - Q_\kappa H(t)Q_\kappa}Q_\kappa \\
&= R(z) - R(z)P_\kappa(P_\kappa R(z)P_\kappa)^{-1}P_\kappa R(z) \qquad (67)
\end{aligned}$$

$$\begin{aligned}
\tilde{R}_0(z) &= Q_\kappa \frac{1}{z - Q_\kappa H Q_\kappa}Q_\kappa \\
&= \lim_{\lambda \to \infty}(z - H - \lambda W)^{-1} \\
&= R_0(z) - R_0(z)P_\kappa(P_\kappa R_0(z)P_\kappa)^{-1}P_\kappa R_0(z). \qquad (68)
\end{aligned}$$

Therefore, we have

$$R_{Q_\kappa}(z) = \tilde{R}_0(z) + \tilde{R}_0(z)V(t)\tilde{R}_0(z) + \tilde{R}_0(z)V(t)\tilde{R}_0(z)V(t)\tilde{R}_0(z) + \cdots. \qquad (69)$$

Substituting equation (69) into (62) gives

$$\begin{aligned}
\psi_\kappa(z) = P_\kappa H P_\kappa &+ P_\kappa V(t)(\tilde{R}_0(z) + \tilde{R}_0(z)V(t)\tilde{R}_0(z) \\
&+ \tilde{R}_0(z)V(t)\tilde{R}_0(z)V(t)\tilde{R}_0(z) + \cdots)V(t)P_\kappa. \qquad (70)
\end{aligned}$$

Hence we obtain the diagonal iterations from equation (70):

$$\begin{aligned}
(Z_\kappa)^{[m]}(t) = (Z_\kappa)^{[0]}(t) \\
+ \sum_{l=0}^{m-1}\langle\varphi_\kappa|V(t)\tilde{R}_0((Z_\kappa)^{[m-1]})[V(t)\tilde{R}_0((Z_\kappa)^{[m-1]})]^l V(t)|\varphi_\kappa\rangle. \qquad (71)
\end{aligned}$$

Therefore the spectral decomposition for the total collision operator is

$$\Theta(t) = \sum_\kappa \psi_\kappa(Z_\kappa(t))|\varphi_\kappa\rangle\langle\varphi_\kappa| = \sum_\kappa Z_\kappa(t)|\varphi_\kappa\rangle\langle\varphi_\kappa|. \qquad (72)$$

From equations (51)–(53*b*), we immediately obtain the spectral decomposition of the Hamiltonian by

$$H(t) = \sum_\kappa Z_\kappa(t)|F_\kappa(t)\rangle\langle\tilde{F}_\kappa(t)| \qquad (73)$$

where the right eigenvectors are

$$|F_\kappa(t)\rangle = \sqrt{\eta_\kappa(t)}(P_\kappa + \tilde{R}_\kappa(Z_\kappa(t))V(t))|\varphi_\kappa\rangle \qquad (74a)$$

and the left eigenvectors are

$$\langle\tilde{F}_\kappa(t)| = \langle\varphi_\kappa|(P_\kappa + V(t)\tilde{R}_\kappa(Z_\kappa(t)))\sqrt{\eta_\kappa(t)} \qquad (74b)$$

and the normalization factor is

$$\sqrt{\eta_\kappa(t)} = (\langle\varphi_\kappa|(P_\kappa + V(t)\tilde{R}_\kappa^2(Z_\kappa(t))V(t))|\varphi_\kappa\rangle)^{-1}. \qquad (74c)$$

In summary, the key to obtaining the spectral decomposition of the Hamiltonian operator $H(t)$ or H is to solve the eigenvalue problem for the collision operator or for the global collision operator, and then to use the intertwining relation, through the similarity operator, to obtain the spectral decomposition.

For the fractal lattice with a self-similar geometric structure, we first solve the eigenvalue problem of the collision operator $\Psi(z)$ which is restricted to a finite subspace of the Hilbert space spanned by eigenvectors of the sites of sublattice 1; then, by using the self-similarity of the system, we can obtain the recursive relations for the eigenvalues and eigenvectors between the two sublattices; finally, we use the intertwining relation between the collision operator and the original Hamiltonian operator, to obtain the spectral decomposition for the original Hamiltonian operator. In the following section, we provide an example to illustrate our approach.

6. The spectral structure of the artificial Sierpinski gasket

We consider the placing of quantum dots [3, 14–16], each confining a single electron, on the sites of the Sierpinski gasket. The inter-electron spaces are sufficiently close on this fractal lattice, that only the nearest-neighbour electrons have an appreciable overlap between their wavefunctions, the spin–spin interaction only influencing nearest neighbours. The Hamiltonian operator of this system is assumed to be given by

$$H = -\beta \sum_{j,k} (|j\rangle\langle k| + |k\rangle\langle j|) \tag{75}$$

where $|j\rangle$, $\langle k|$ represent the eigenstates of the nearest-neighbour neurons j, k, respectively.

We divide the system into two sublattices, sublattice 1 including a large triangular form, while sublattice 2 includes a small triangular form. We start from the first-order lattice and denote $j^{(\nu)}$, $k^{(\nu)} = 1^{(\nu)}, 2^{(\nu)}, 3^{(\nu)}$, $\nu = 1, 2$ to represent sublattice 1 and sublattice 2 of the first-order lattice, respectively (see figure 1). Then the matrix of $\Psi_M^{(1)}$ with respect to the vectors $\{j^{(1)}, k^{(1)}\}$ is given by:

$$H^{12}(ZI - H^{22})^{-1}H^{21} = \begin{pmatrix} \frac{2\varepsilon\beta}{(\varepsilon+2)(\varepsilon-1)} & \frac{(2-\varepsilon)\beta}{(\varepsilon+2)(\varepsilon-1)} & \frac{(2-\varepsilon)\beta}{(\varepsilon+2)(\varepsilon-1)} \\ \frac{(2-\varepsilon)\beta}{(\varepsilon+2)(\varepsilon-1)} & \frac{2\varepsilon\beta}{(\varepsilon+2)(\varepsilon-1)} & \frac{(2-\varepsilon)\beta}{(\varepsilon+2)(\varepsilon-1)} \\ \frac{(2-\varepsilon)\beta}{(\varepsilon+2)(\varepsilon-1)} & \frac{(2-\varepsilon)\beta}{(\varepsilon+2)(\varepsilon-1)} & \frac{2\varepsilon\beta}{(\varepsilon+2)(\varepsilon-1)} \end{pmatrix} \tag{76}$$

where

$$\varepsilon \equiv \frac{Z}{\beta}. \tag{77}$$

The eigenvalues and eigenvectors for this matrix are:

$$\varepsilon_1^{(1)} \leftrightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{78a}$$

and

$$\varepsilon_2^{(1)} \leftrightarrow \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{78b}$$

where

$$\varepsilon_1^{(1)} = \frac{1}{3}\sqrt[3]{(44 + 3\sqrt{177})} + \frac{7}{3\sqrt[3]{(44 + 3\sqrt{177})}} - \frac{1}{3} \tag{79a}$$
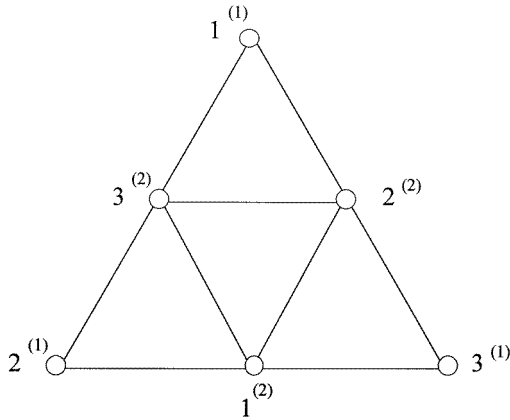
Figure 1. First-order Sierpinski gasket lattice.

$$\varepsilon_2^{(1)} = -\frac{1}{12}\sqrt[3]{(-404 + 12\mathrm{i}\sqrt{687})} - \frac{16}{3\sqrt[3]{(-404 + 12\mathrm{i}\sqrt{687})}} - \frac{1}{3}$$

$$+ \frac{1}{2}\mathrm{i}\sqrt{3}\left(\frac{1}{6}\sqrt[3]{(-404 + 12\mathrm{i}\sqrt{687})} - \frac{32}{3\sqrt[3]{(-404 + 12\mathrm{i}\sqrt{687})}}\right). \tag{79b}$$

These give the same eigenvalues of the operator $\Psi^{(1)}(z)$ and its eigenvectors which are combined by:

$$|\varphi_1^{(1)}\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle) \tag{80a}$$

$$|\varphi_2^{(1)}\rangle = \frac{1}{\sqrt{2}}(-|1\rangle + |2\rangle) \tag{80b}$$

$$|\varphi_3^{(1)}\rangle = \frac{1}{\sqrt{2}}(-|1\rangle + |3\rangle). \tag{80c}$$

The recursive formula of the eigenvalues between sublattice 1, and sublattice 2 for the nth-order and $(n-1)$th-order lattice can be inferred by induction using equation (35):

$$\varepsilon' = \varepsilon\frac{\varepsilon^2 + \varepsilon - 4}{2 - \varepsilon}. \tag{81}$$

Then, from equation (81), the recursive formula of the eigenvalues between the nth-order and $(n-1)$th-order lattices is given by

$$\varepsilon^{(n-1)} = \frac{1}{6}\sqrt[3]{(252\varepsilon^{(n)} - 152 + 12\sqrt{(12(\varepsilon^{(n)})^3 + 285(\varepsilon^{(n)})^2 + 144\varepsilon^{(n)} - 816)})}$$

$$- 6\frac{\frac{1}{3}\varepsilon^{(n)} - \frac{13}{9}}{\sqrt[3]{(252\varepsilon^{(n)} - 152 + 12\sqrt{(12(\varepsilon^{(n)})^3 + 285(\varepsilon^{(n)})^2 + 144\varepsilon^{(n)} - 816)})}} - \frac{1}{3} \tag{82}$$

where $-4 < \varepsilon^{(n)} < 4$, and the diagonal elements and off-diagonal elements of the matrix $\Psi_M^{(1)}$ are

$$\alpha' = \frac{2\varepsilon\beta}{(\varepsilon + 2)(\varepsilon - 1)} \tag{83a}$$

$$\beta' = \frac{(2 - \varepsilon)\beta}{(\varepsilon + 2)(\varepsilon - 1)} \tag{83b}$$

and ε is defined by

$$\varepsilon' \equiv \frac{Z - \alpha'}{\beta'}. \tag{83c}$$

Furthermore, in terms of equation (39), we have the recursive formula of eigenvectors between sublattice 1 and sublattice 2 given by:

$$\varphi_j^{(2)} = \frac{1}{(\varepsilon + 2)(\varepsilon - 1)} \sum_{j',k=1}^{3} -(ZI - H^{(22)})_{jj'}^{-1}(H^{(21)})_{j'k}\varphi_k^{(1)} \tag{84}$$

which provides the recursive formula for nth-order and $(n-1)$th-order lattices:

$$\varphi_k^{(n)} = \frac{1}{(\varepsilon + 2)(\varepsilon - 1)} \sum_{j=1}^{3} -[\varepsilon - (\varepsilon + 2)\delta_{kj}]\varphi_j^{(n-1)}. \tag{85}$$

From the recursive formula of the spectrum (i.e. equation (81)), we can see that the zero-permitted energy band I_0 can be obtained by

$$-4 \leqslant \varepsilon_0 \frac{\varepsilon_0^2 + \varepsilon_0 - 4}{2 - \varepsilon_0} \leqslant 4 \tag{86}$$

$$I_0 : a_0 \leqslant \varepsilon_0 \leqslant b_0 \qquad \varepsilon_0 \neq 2 \tag{87}$$

where

$$a_0 \equiv -\frac{1}{3}\sqrt[3]{(145 + 30\sqrt{6})} - \frac{25}{3\sqrt[3]{(145 + 30\sqrt{6})}} - \frac{1}{3} \tag{88a}$$

$$b_0 \equiv \frac{1}{3}\sqrt[3]{(107 + 6\sqrt{318})} + \frac{1}{3\sqrt[3]{(107 + 6\sqrt{318})}} - \frac{1}{3}. \tag{88b}$$

Repeating the same calculation we can find I_1, I_2, and so forth. The tendency of a_k, b_k is that they are restricted to the interval $[a_0, b_0]$ and move closer to each other with increasing k, until arriving at the fixed points: $a_k = -1 - \sqrt{7}$, $b_k = -1 + \sqrt{7}$. Therefore, the permitted energy band of the infinite Sierpinski gasket is $[-1 - \sqrt{7}, -1 + \sqrt{7}]$. But this is not always the case. For example, if we assume that the recursive formula of the spectrum is given by

$$\varepsilon' = -\varepsilon(\varepsilon + 3) \tag{89}$$

corresponding to the two opposite Sierpinski gaskets connected together with a periodic boundary condition (see figure 2) [18], we have the forbidden energy bands:

$$I_0 : \varepsilon_0 > 1, \varepsilon_0 < -4$$

$$I_1 : -\frac{3 + \sqrt{5}}{2} < \varepsilon_1 < \frac{3 - \sqrt{5}}{2}$$

$$I_2 : -\frac{3}{2} - \frac{1}{2}\sqrt{(15 + 2\sqrt{5})} < \varepsilon < -\frac{3}{2} - \frac{1}{2}\sqrt{(15 - 2\sqrt{5})}$$

$$-\frac{3}{2} + \frac{1}{2}\sqrt{(15 - 2\sqrt{5})} < \varepsilon < -\frac{3}{2} + \frac{1}{2}\sqrt{(15 + 2\sqrt{5})}$$

$$\dots \dots \tag{90}$$

Continuing this process we can find that I_k will cover the interval $[-4, 1]$, for $k \to \infty$ [18]. The permitted values of energy form a Cantor set with zero measure, which is different from the previous case. Therefore, choosing suitable boundary conditions is crucial for designing a fractal network that reflects expected properties.
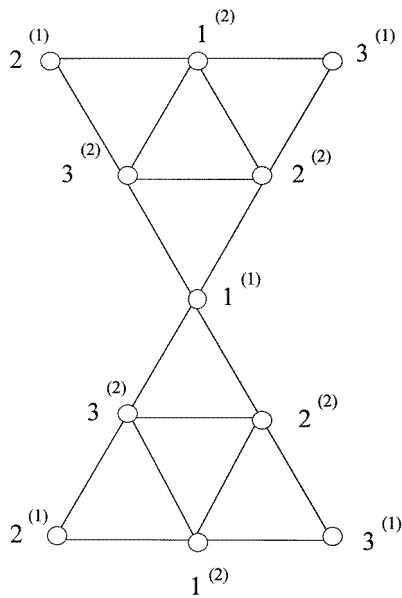
Figure 2. First-order 'double-opposite' Sierpinski gasket lattice.

7. Conclusions

We have presented a general formulation to deal with the spectral decomposition of the Hamiltonian operator for different sorts of confinement fractal network by combining subdynamics and reduced sublattice approaches. We connected the collision operator or the intermediate operator with the so-called effective Hamiltonian and have pointed out that the main method to solve the eigenvalue problem for the original Hamiltonian operator is to solve for the eigenvalues of the collision operator or its matrix representation by reducing one sublattice. Once we have solved the eigenvalue problem of the collision operator for such a first-order lattice, we can use self-similarity of the system to deduce the recursive formulae of the eigenvalues and eigenvectors of the collision operator for an arbitrary-order lattice. Finally, we can obtain the spectral decomposition of the original Hamiltonian operator through the intertwining relation between the original Hamiltonian operator and the intermediate operator.

As a demonstration of our approach, we calculated the eigenvalues and eigenvectors of the Hamiltonian operator for the Sierpinski gaskets based on our formulation, which shows that the formulation is useful and powerful for solving the eigenvalue problem for a QFN.

Acknowledgment

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